Exercises EE

1 warmup

Consider the two-qubit state

$$|\Psi\rangle = \frac{1}{\sqrt{2}}|\uparrow\rangle_A \left(\frac{1}{2}|\uparrow\rangle_B + \frac{\sqrt{3}}{2}|\downarrow\rangle_B\right) + \frac{1}{\sqrt{2}}|\downarrow\rangle_A \left(\frac{\sqrt{3}}{2}|\uparrow\rangle_B + \frac{1}{2}|\downarrow\rangle_B\right) \tag{1}$$

Compute the reduced density matrices ρ_A and ρ_B and find the Schmidt decomposition of $|\Psi\rangle$. Compute the n-th Renyi entropy

$$S_A^n = \frac{1}{1-n}\log \mathrm{Tr}\rho_A^n \tag{2}$$

2 Negativity

Entanglement negativity is yet another interesting quantity. In some sense mutual information measures correlations between subsystems but not quite entanglement, and entanglement negativity has been found to be useful alternative to quantify the entanglement between subsystems. To define it we write the density matrix of the AB system as

$$\rho = \sum_{a,b,a',b'} \rho_{ab}^{a'b'} |a\rangle_A |b\rangle_B \langle b'|_B \langle a'|_B \tag{3}$$

and define the partial transpose as

$$\rho^T = \sum_{a,b,a',b'} \rho_{ab'}^{a'b} |a\rangle_A |b\rangle_B \langle b'|_B \langle a'|_B. \tag{4}$$

Notice that we have taken the transpose of the B-indices but have not changed anything else. Now the definition of entanglement negativity is

$$\mathcal{E}_{A,B} = \log \mathrm{Tr}|\rho^T| \equiv \sum_i \log|\lambda_i| \tag{5}$$

where λ_i are the eigenvalues of ρ^T.

- Denote by $\mathcal{E}^n_{A,B} = \mathrm{Tr}(\rho^T)^n$. Show that

$$n = \text{even} \qquad \mathcal{E}^n_{A,B} = \sum_{\lambda_i > 0} |\lambda_i|^n + \sum_{\lambda_i < 0} |\lambda_i|^n \tag{6}$$

$$n = \text{odd} \qquad \mathcal{E}^n_{A,B} = \sum_{\lambda_i > 0} |\lambda_i|^n - \sum_{\lambda_i < 0} |\lambda_i|^n \tag{7}$$

- Explain why entanglement negativity is the analytic continuation of $\log \mathcal{E}^n_{A,B}$ *with only n even* to $n = 1$.

- Suppose $\rho = |\psi\rangle\langle\psi|$ is a pure state. Write ψ in a Schmidt decomposition $|\psi\rangle = \sum_i \mu_i |i\rangle_A |\tilde{i}\rangle_B$. Give an explicit expression for ρ^T and compute $\mathcal{E}^n_{A,B}$ for even n.

- Analytically continue the result to $n = 1$ to show that for pure states the entanglement negativity equals the Renyi with $n = 1/2$:

$$\mathcal{E}_{A,B} = S_A^{1/2} \tag{8}$$

3 Genus of replica manifold

If we compute the n-th Renyi for a conformal field theory in $1+1$ dimensions for a subsystem consisting of N disjoint intervals, the manifold that we get by gluing together n copies of the original Euclidean manifold along the N intervals is quite complicated. Try to convince yourself that the resulting manifold is a Riemann surface of genus $g = (n-1)(N-1)$. (the genus of a Riemann surfaces is roughly speaking the number of holes). (for a picture with $n = 3$ and $N = 2$ see e.g. figure 1 in https://arxiv.org/pdf/0905.2069.pdf).

4 Holographic EE

The metric for a constant time slice of AdS_3 is

$$ds^2 = \frac{dz^2 + dx^2}{z^2}. \tag{9}$$

Compute the length of a geodesic in this geometry between the points $(z, x) = (\epsilon, 0)$ and $(z, x) = (\epsilon, L)$ for small ϵ. Compute the entanglement entropy and compare with the result in the lecture using that the central charge c is related to the Newton constant via $c = 3/2G$.

5 C-theorem in two dimensions

Consider a unitary, Lorentz invariant 2d quantum field theory. We can associate entanglement entropy to an interval of proper spatial length ℓ which gives us a function $S(\ell)$. This interval does not have to be on a particular spatial slice, by Lorentz invariance there is no distinguished slice anyway and the entanglement entropy can only depend on the proper length ℓ. Now define the following four points in the (x,t) plane

$$P = (-a-e, e), \; Q = (-a, 0), \; R = (a, 0), \; S = (a+e, e). \tag{10}$$

Given a pair of space-like separated points, there is a tensor factor in the Hilbert space which one would get by quantizing a theory with a choice of Cauchy surface which goes through the two points, and by restricting to the degrees of freedom localized between the two points.

- Denote by A the subsystem defined by the pair P, R and B the subsystem defined by Q, S. Show that one should associate $A \cap B$ to Q, R and $A \cup B$ to P, S.

- Show that strong subadditivity

$$S(A) + S(B) \geq S(A \cup B) + S(A \cap B) \tag{11}$$

  implies that the function $S(\ell)$ must obey

$$2S(2\sqrt{a^2 + ae}) \geq S(2a) + S(2a + 2e). \tag{12}$$

- Expand this equation to second order in e to show that

$$\partial_\ell(\ell \partial_\ell S(\ell)) \leq 0. \tag{13}$$

The function $C(\ell) = 3\ell\partial_\ell S(\ell)$ is a C-function: For conformal field theories it is equal to a constant, the central charge, and for non-conformally invariant theories it is a function which interpolates monotonically (which is what you just showed) between the central charge of the UV fixed point $c_{UV} = C(0)$ to the central charge of the IR fixed point $c_{IR} = C(\infty)$. It is not identical to the c-function of Zamolodchikov which has similar properties but is defined in a different way using the two point functions of the stress tensor.

6 Firewalls

Suppose we make a black hole by collapsing some matter which initially was in a pure state. After the black hole is created it will start to produce Hawking radiation. Eventually the entire black hole will evaporate away leaving only Hawking radiation behind. When the black hole is there we can choose a Cauchy surface (the sort of spatial surfaces you can use to define proper initial conditions and which you can use to quantize the theory, roughly speaking) which goes through the horizon and on that surface we can restrict to the outside of the black hole and compute the reduced density matrix. This reduced density matrix describes the Hawking radiation outside the black hole. Argue that the entanglement entropy of this reduced density matrix will first increase but eventually decrease in time again and end up begin zero at late times (if you believe in unitarity). The moment where it reaches its maximal value is called the "Page time".

Now let's take a moment well after this Page time and look at the Cauchy slice and define three subsystems: A is the subsystem containing early Hawking radiation, B consists of some late Hawking radiation, and C consists of modes behind the horizon which are correlated with the late Hawking radiation. The modes in B and C are very much like the modes we saw in the Rindler example and in order for the horizon to be a smooth place the systems B and C must by highly entangled as together they must approximately describe a Minkowski vaccum. Since the BC system describes approximately a pure state, $S(BC)$ must be approximately zero. Also, because we are after the page time, $S(AB) < S(A)$. Moreover, $S(B)$ is not zero, because the B system is approximately thermal just as in the Rindler example. Show that the above statements violate strong subadditivity and therefore cannot all hold at the same time. (hint: first argue that $S(ABC) = S(A)$).

This is the essence of the firewall argument: if we assume local effective field theory is valid, that there is nothing special at the horizon and that the theory is unitary we run into a contradiction. One resolution is to give up the notion that there is nothing special happening at the horizon.

7 Temperature of a black hole

Just in case you have never done this: consider the Schwarzschild metric

$$ds^2 = -(1 - 2M/r)dt^2 + (1 - 2M/r)^{-1}dr^2 + r^2 d\Omega^2. \tag{14}$$

Wick rotate t to Euclidean time and find which periodic identification we have to do on Euclidean time to make the Euclidean metric regular. This period is the inverse temperature of the black hole.

8 Quantum error correction

To get an idea how quantum error correction works, read for example the first part of section 3.1 of https://arxiv.org/pdf/1411.7041.pdf and verify the statements made there.

9 Rindler the old fashioned way

We consider a massless scalar in 2d Minkowski and Rindler spacetime. It will be convenient to use light-cone coordinates because in those the solutions of the field equations are very simple. We denote by $\bar{u}, \bar{v}$ the lightcone coordinates of Minkoswki spacetime and by u, v lightcone coordinates of Rindler spacetime. The relation between them is

$$\bar{u} = -e^{-u}, \qquad \bar{v} = e^{v} \tag{15}$$

and the metric for these coordinates is

$$ds^2 = -d\bar{u}d\bar{v} = -e^{v-u}dudv \tag{16}$$

- Write the action for a massless scalar field in each coordinate system and find the most general solution of the field equations.

The general solution of the field equations is quite similar in both coordinate systems. For Minkowski spacetime we write

$$\phi(\bar{u}, \bar{v}) = \int_0^\infty \frac{d\omega}{\sqrt{2\pi}} \frac{1}{\sqrt{2\omega}} \left[e^{-i\omega\bar{u}} \hat{a}^-_\omega + e^{i\omega\bar{u}} \hat{a}^+_\omega + e^{-i\omega\bar{v}} \hat{a}^-_{-\omega} + e^{i\omega\bar{v}} \hat{a}^+_{-\omega} \right] \tag{17}$$

Similarly, in Rindler we have (where now Ω is the frequency to distinguish it from ω)

$$\phi(u,v) = \int_0^\infty \frac{d\Omega}{\sqrt{2\pi}} \frac{1}{\sqrt{2\Omega}} \left[e^{-i\Omega u} \hat{b}_\Omega^- + e^{i\Omega u} \hat{b}_\Omega^+ + e^{-i\Omega v} \hat{b}_{-\Omega}^- + e^{i\Omega v} \hat{b}_{-\Omega}^+ \right] \tag{18}$$

The Minkowski vacuum is the state which is annihilated by the annihilation operators $\hat{a}^-$. Similarly, the Rindler vacuum is annihilated by $\hat{b}^-$. The creation and annihilation operators obey the usual commutation relations.

c) Show that for $\Omega > 0$

$$\hat{b}_\Omega^- = \int_0^\infty d\omega [\alpha_{\omega\Omega} \hat{a}_\omega^- + \beta_{\omega\Omega} \hat{a}_\omega^+] \tag{19}$$

where

$$\alpha_{\omega\Omega} = \sqrt{\frac{\Omega}{\omega}} F(\omega, \Omega), \qquad \beta_{\omega\Omega} = \sqrt{\frac{\Omega}{\omega}} F(-\omega, \Omega) \tag{20}$$

and

$$F(\omega, \Omega) = \int_{-\infty}^\infty \frac{du}{2\pi} \exp\left[i\Omega u + i\omega e^{-u} \right] \tag{21}$$

d) Show that for a general Bogolyubov transformation

$$\hat{b}_\Omega^- = \int d\omega [\alpha_{\omega\Omega} \hat{a}_\omega^- + \beta_{\omega\Omega} \hat{a}_\omega^+] \tag{22}$$

the canonical commutation relations imply

$$\int d\omega (\alpha_{\omega\Omega} \alpha^*_{\omega\Omega'} - \beta_{\omega\Omega} \beta^*_{\omega\Omega'}) = \delta(\Omega - \Omega') \tag{23}$$

e) Show that

$$F(\omega, \Omega) = F(-\omega, \Omega) e^{\pi\Omega} \tag{24}$$

using contour deformation in the definition of $F(\omega, \Omega)$.

f) Combine (20),(23) and (24) to show that

$$\int_0^\infty d\omega \frac{\sqrt{\Omega\Omega'}}{\omega} F(-\omega, \Omega) F^*(-\omega, \Omega') = \frac{\delta(\Omega - \Omega')}{e^{2\pi\Omega} - 1}. \tag{25}$$

g) Now combine everything to compute the expectation value of Rindler particle number in the Minkowski vacuum

$$\langle \hat{N}_\Omega \rangle = \langle 0 | \hat{b}_\Omega^+ \hat{b}_\Omega^- | 0 \rangle = \int d\omega |\beta_{\omega\Omega}|^2 = \frac{1}{e^{2\pi\Omega} - 1} \delta(0). \tag{26}$$

The final result (26) shows that the expectation value of particle number is the finite temperature Bose-Einstein result, the $\delta(0)$ is a volume factor which reflects the fact that we are in infinite volume, if you remove that you get the density of particles as seen by a Rindler observer is purely thermal with temperature $T = \frac{1}{2\pi}$ as in the lecture.

Notice that at the end we did not need that many details of the theory. What was essential was the result (24).

Yet another way to get the finite temperature result is to compute the response of an accelerated detector in Minkowski space-time. This involves an integral of the positive frequency Wightman function (Greens function) along the detector trajectory and one also finds a thermal bath (see e.g. Birrell and Davies)

h)** You now have all the ingredients in place to compute the expectation value of the Minkowski space-time energy-momentum tensor in the Rindler ground state (or in an excited Rindler state). You will find that it diverges at the Rindler horizon. Smooth Minkowski space-time only emerges in a mixed state (which results from the entanglement between the two Rindler regions and tracing over one of the two).